The Laplace-Beltrami operator on surfaces with axial symmetry

1998 J. Phys. A: Math. Gen. 31 4289

(http://iopscience.iop.org/0305-4470/31/18/017)

J. Phys. A: Math. Gen. **31** (1998) 4289–4300. Printed in the UK PII: S0305-4470(98)82447-5

The Laplace–Beltrami operator on surfaces with axial symmetry

Emil Prodan†

University of Houston, 4800 Calhoun Rd, Houston, TX 77204-5508, USA

Received 24 February 1998

Abstract. A solution for the mathematical problem of functional calculus with the Laplace–Beltrami operator on surfaces with axial symmetry is found. A quantitative analysis of the spectrum is presented.

1. Introduction

The physical situation which has initiated this research is that of a dielectric particle with electrical charges on its surface, placed in electric field. Here, the diffusion equation of the charges is coupled with the Maxwell equations. There is an analytical solution of this system of equation [1] which involves some functional calculus with operators, in particular with the Laplace–Beltrami operator defined on the surface of that particle. We can imagine many other physical situations described by a complicated system of equations where the Laplace–Beltrami operator is implicated (e.g. that of the acoustic wave scattering on particles with membrane, etc). As before, one can find a compact solution by using functional calculus. However, these solutions are not complete because, at this level, all is formal. We must have an effective procedure to calculate the expressions which involve operators. One can try to compute the matrices of those operators in some orthonormal basis and to transform the problem into an algebraic one. The practical problem is that one can compute only a finite number of matrix elements and this can lead to serious problems when unbounded operators are implicated. If we choose an inappropriate basis, it is possible that the expressions, calculated with truncated matrices, will not converge at the correct result.

In this paper we will find an orthonormal basis in the space of square integrable functions defined on a surface with axial symmetry such that the truncated matrices of the Laplace–Beltrami operator converge in the norm resolvent sense. Then, according to [5] we can use these truncated matrices in functional calculus.

2. The result

Let $\mathbb{M}$ be a C^∞ closed two-dimensional surface which in the spherical coordinates $\{r, \theta, \phi\}$ relative to a three-orthogonal system of axes is defined by the equation $r = r(\theta)$. We consider that all necessary conditions having a C^∞ surface are fulfilled. Let this surface be

† E-mail address: prodan@shasta.phys.uh.edu

equipped with the metric which is induced by the embedding in $\mathbb{R}^3$ and let $x_0 \in \mathbb{M}$ be the point defined by $\theta = 0$. Relative to this point, the normal coordinates $\{\lambda, \varphi\}$ are defined by

$$\begin{cases} \lambda(x(\theta,\phi)) = \mathrm{d}(x(\theta,\phi), x_0) = \displaystyle\int_0^\theta \mathrm{d}t\,\sqrt{r(t)^2 + r'(t)^2} \\ \varphi = \phi \end{cases}$$

which parametrize the entire surface, without the points $\theta = 0, \pi$. We define $R = \lambda(x(\pi))/\pi$ and the new coordinates: $\{\vartheta = \lambda/R, \varphi\}$. In these coordinates, the metric form is

$$g(\vartheta,\varphi) = \begin{pmatrix} R^2 & 0 \\ 0 & r(\theta(\vartheta))^2 \sin(\theta(\vartheta))^2 \end{pmatrix}.$$

Proposition 1. The set of C^∞ functions:

$$\mathcal{Y}_{lm} : \mathbb{M} \to \mathbb{C} \qquad \mathcal{Y}_{lm}(\vartheta,\phi) = \sqrt{\frac{R\sin\vartheta}{r(\theta(\vartheta))\sin\theta(\vartheta)}}\frac{Y_{lm}(\vartheta,\varphi)}{R} \qquad m \in \mathbb{Z},\ l \geqslant |m|$$

is an orthonormal basis in $L_2(\mathbb{M}, \mu_g)$, where Y_{lm} represent the spherical harmonics and μ_g is the measure induced on $\mathbb{M}$ by the metric g.

Proof. The orthonormality:

$$\begin{aligned} \langle \mathcal{Y}_{lm}, \mathcal{Y}_{l'm'}\rangle &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\,\sqrt{\det g}\cdot \mathcal{Y}_{lm}(\vartheta,\varphi)\mathcal{Y}^*_{l'm'}(\vartheta,\varphi) \\ &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, Rr(\theta)\sin\theta \cdot \frac{R\sin\vartheta}{r\sin\theta}\frac{\mathcal{Y}_{lm}(\vartheta,\varphi)}{R}\frac{\mathcal{Y}^*_{l'm'}(\vartheta,\varphi)}{R} \\ &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\,\sin\vartheta\cdot \mathcal{Y}_{lm}(\vartheta,\varphi)\mathcal{Y}^*_{lm}(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'}. \end{aligned}$$

The completeness:

$$\begin{aligned} &\int_0^\pi \mathrm{d}\vartheta' \int_0^{2\pi} \mathrm{d}\varphi'\,\sqrt{\det g}\cdot \sum_{l,m} \mathcal{Y}_{lm}(\vartheta,\varphi)\mathcal{Y}^*_{l'm'}(\vartheta',\varphi') f(\vartheta',\varphi') \\ &\quad = \int_0^\pi \mathrm{d}\vartheta' \int_0^{2\pi} \mathrm{d}\varphi\, Rr(\theta')\sin(\theta') \\ &\qquad \times \sum_{l,m} \sqrt{\frac{R\sin\vartheta}{r(\theta)\sin\theta}}\sqrt{\frac{R\sin\vartheta'}{r(\theta')\sin\theta'}}\frac{Y_{lm}(\vartheta,\varphi)}{R}\frac{Y^*_{lm}(\vartheta',\varphi')}{R} f(\vartheta',\varphi') \\ &\quad = \sqrt{\frac{R\sin\vartheta}{r(\theta)\sin\theta}} \int_0^\pi \mathrm{d}\vartheta' \int_0^{2\pi} \mathrm{d}\varphi' \sum_{l,m} Y_{lm}(\vartheta,\varphi) Y^*_{lm}(\vartheta',\varphi')\sqrt{\frac{r(\theta')\sin\theta'}{R\sin(\vartheta')}} f(\vartheta',\varphi') \\ &\quad = f(\vartheta,\varphi) \end{aligned}$$

because $\sqrt{\frac{r(\theta)\sin\theta}{R\sin(\vartheta)}} f(\vartheta,\varphi)$ is in $L_2(\mathbb{M},\mu_g)$ if $f \in L_2(\mathbb{M},\mu_g)$. □

For a fixed m, let $\boldsymbol{S}_m$ be the Hilbert subspace spanned by $\{\mathcal{Y}_{lm}\}_{l\geqslant|m|}$, which is invarianted by the Laplace–Beltrami operator. In the following, we will consider the restriction of this operator at a $\boldsymbol{S}_m$ subspace, $\Delta^{(m)} = \Delta|_{\boldsymbol{S}_m}$. Let $P_k^{(m)}, k \geqslant |m|$, be the projection on the subspace spanned by the vectors $\mathcal{Y}_{|m|m}, \ldots, \mathcal{Y}_{km}$. Our main result is shown below.

Theorem 2. The sequence of operators

$$\{P_k^{(m)}[P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z]^{-1}\}_{k \geqslant |m|}$$

converges in norm topology at the operator $[\Delta^{(m)} - z]^{-1}$, for any $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$.

Proof. We have successively:

$$\begin{aligned}
P_k^{(m)} & \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \\
&= P_k^{(m)} \left[\frac{1}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right] - (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - z} \\
&= \frac{1}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} P_k^{(m)} \Delta^{(m)} (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - z} - (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - z} \\
&= \frac{z}{P_k^{(m)} \Delta^{(m)} P_k^{(m)} - z} \left[I + \frac{1}{z} P_k^{(m)} \Delta^{(m)} (I - P_k^{(m)}) \right] (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - z}.
\end{aligned}$$

Without loss of generality we can choose $z = \mathrm{i}\omega$, $\omega \in \mathbb{R}$, $\omega \neq 0$. Thus:

$$\begin{aligned}
&\left\| P_k^{(m)} \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right\| \\
&\qquad \leqslant \left(1 + \frac{1}{\omega} \| P_k^{(m)} \Delta^{(m)} (I - P_k^{(m)}) \| \right) \cdot \left\| (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - \mathrm{i}\omega} \right\|.
\end{aligned}$$

Lemma 3. For $l, k \in \mathbb{N}$, $l, k \geqslant |m|$, with $l \neq k$:

$$\langle \mathcal{Y}_{lm}, \Delta^{(m)} \mathcal{Y}_{km} \rangle = \langle \mathcal{Y}_{lm}, h \cdot \mathcal{Y}_{km} \rangle$$

where h is at least a C^0 function and M. The operators $P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)})$ are bounded and their norms satisfy:

$$\| P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)}) \| \leqslant \| h \|_\infty.$$

It follows that

$$\left\| P_k^{(m)} \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - z} - \frac{1}{\Delta^{(m)} - z} \right\| \leqslant \left(1 + \frac{\|h\|_\infty}{\omega} \right) \cdot \left\| (I - P_k^{(m)}) \frac{1}{\Delta^{(m)} - \mathrm{i}\omega} \right\|.$$

To evaluate the last norm, we use the following lemma.

Lemma 4. Let $s(\vartheta)$ be the quantity $\sqrt{\frac{r \sin\theta}{R \sin\vartheta}}$. If $\lambda_{|m|}^{(m)}, \ldots, \lambda_n^{(m)}, \ldots$ are the ordered eigenvalues of $\Delta^{(m)}$ and $\upsilon_{|m|}^{(m)}, \ldots, \upsilon_n^{(m)}, \ldots$ are the corresponding eigenvectors, then for any $m \in \mathbb{Z}$ and $l \geqslant |m|, l \geqslant 1$ and $n \geqslant |m|$

$$|\langle \mathcal{Y}_{lm} | \upsilon_n^{(m)} \rangle| \leqslant \frac{Rc \left[\|\mathrm{d}s\|_\infty + c\sqrt{\lambda_n^{(m)}} \right]}{\sqrt{l(l+1)}}$$

and for any $m \in \mathbb{Z}$, $l \geqslant |m|$, and $n \geqslant |m|$, $n \geqslant 1$

$$|\langle \mathcal{Y}_{lm} | \upsilon_n^{(m)} \rangle| \leqslant \frac{c \left[\|\mathrm{d}s^{-1}\|_\infty + \frac{c}{R}\sqrt{l(l+1)} \right]}{\sqrt{\lambda_n^{(m)}}}.$$

Moreover:

$$\frac{1}{c^2}\frac{l(l+1)}{R^2} \leqslant \lambda_l^{(m)} \leqslant c^2\frac{l(l+1)}{R^2}$$

where $c = [\max\{\|s^2\|_\infty, \|s^{-2}\|_\infty\}]^{1/2}$.

Now, let $\upsilon \in L_2(\mathbb{M}, \mu_g)$, $\upsilon = \sum_{n\geqslant|m|} a_n \cdot \upsilon_n^{(m)}$. Then

$$\begin{aligned}
\left\|(I - P_k^{(m)})\frac{1}{\Delta^{(m)} - \mathrm{i}\omega}\upsilon\right\|^2 &= \left\|\sum_{l\geqslant k+1}\sum_{n\geqslant|m|} a_n \cdot \frac{\langle \mathcal{Y}_{lm}, \upsilon_n^{(m)}\rangle}{\lambda_n^{(m)} - \mathrm{i}\omega}\mathcal{Y}_{lm}\right\|^2 \\
&= \sum_{l\geqslant k+1}\left|\sum_{n\geqslant|m|} a_n \cdot \frac{\langle \mathcal{Y}_{lm}, \upsilon_n^{(m)}\rangle}{\lambda_n^{(m)} - \mathrm{i}\omega}\right|^2 \leqslant \sum_{l\geqslant k+1}\sum_{n\geqslant|m|}|a_n|^2 \cdot \sum_{n\geqslant|m|}\left|\frac{\langle \mathcal{Y}_{lm}, \upsilon_n^{(m)}\rangle}{\lambda_n^{(m)} - \mathrm{i}\omega}\right|^2 \\
&\geqslant \|\upsilon\|^2 \sum_{l\geqslant k+1}\sum_{n\geqslant|m|}\left|\frac{cR\left[\|\mathrm{d}s\|_\infty + c\sqrt{\lambda_n^{(m)}}\right]}{\sqrt{l(l+1)}(\lambda_n^{(m)} - \mathrm{i}\omega)}\right|^2 \\
&= (cR)^2\frac{\|\upsilon\|^2}{k+1}\sum_{n\geqslant|m|}\left\|\frac{\left[\|\mathrm{d}s\|_\infty + c\sqrt{\lambda_n^{(m)}}\right]}{\lambda_n^{(m)} - \mathrm{i}\omega}\right\|^2.
\end{aligned}$$

For $|m| > 0$,

$$\begin{aligned}
\left\|(I - P_k^{(m)})\frac{1}{\Delta^{(m)} - \mathrm{i}\omega}\upsilon\right\|^2 &\leqslant (cR)^2\frac{\|\upsilon\|^2}{k+1}\sum_{n\geqslant|m|}\frac{1}{\lambda_n^{(m)}}\frac{\left[c + \frac{\|\mathrm{d}s\|_\infty}{\sqrt{\lambda_n^{(m)}}}\right]^2}{1 + \frac{\omega^2}{\lambda_n^{(m)2}}} \\
&\leqslant (cR)^2\frac{\|\upsilon\|^2}{k+1}\left[c + \frac{\|\mathrm{d}s\|_\infty}{\sqrt{\lambda_{|m|}^{(m)}}}\right]^2\sum_{n\geqslant|m|}\frac{1}{\lambda_n^{(m)}}.
\end{aligned}$$

Finally

$$\left\|(I - P_k^{(m)})\frac{1}{\Delta^{(m)} - \mathrm{i}\omega}\right\| \leqslant \frac{(cR)^2}{\sqrt{(k+1)|m|}}\left[c + \frac{cR\|\mathrm{d}s\|_\infty}{\sqrt{|m|(|m|+1)}}\right].$$

For $|m| = 0$,

$$\begin{aligned}
\left\|(I - P_k^{(0)})\frac{1}{\Delta^{(0)} - \mathrm{i}\omega}\upsilon\right\|^2 &\leqslant \frac{\|\upsilon\|^2}{k+1}\left(\frac{(cR)^2\|\mathrm{d}s\|_\infty^2}{\omega^2}\sum_{n\geqslant1}(cR)^2\frac{1}{\lambda_n^{(0)}}\frac{\left[c + \frac{\|\mathrm{d}s\|_\infty}{\sqrt{\lambda_n^{(0)}}}\right]}{1 + \frac{\omega^2}{\lambda_n^{(0)2}}}\right) \\
&\leqslant (cR)^2\frac{\|\upsilon\|^2}{k+1}\left(\frac{\|\mathrm{d}s\|_\infty^2}{\omega^2} + (cR)^2\left[c + \frac{cR\|\mathrm{d}s\|_\infty}{\sqrt{2}}\right]^2\right)
\end{aligned}$$

thus:

$$\left\|(I - P_k^{(0)})\frac{1}{\Delta^{(0)} - \mathrm{i}\omega}\right\| \leqslant \frac{(cR)^2}{\sqrt{(k+1)}}\sqrt{\frac{\|\mathrm{d}s\|_\infty^2}{(cR)^2\omega^2} + \left[c + \frac{cR\|\mathrm{d}s\|_\infty}{\sqrt{2}}\right]^2}.$$

Having that $\|\mathrm{d}s\|_\infty = \frac{1}{R}\|\frac{\partial s}{\partial\vartheta}\|_\infty = \frac{1}{R}\|s'\|_\infty$ we can conclude:

$$\left\| P_k^{(m)} \frac{1}{P_k^{(m)} \circ \Delta^{(m)} \circ P_k^{(m)} - \mathrm{i}\omega} - \frac{1}{\Delta^{(m)} - \mathrm{i}\omega} \right\| \leqslant \begin{cases} \left(1 + \frac{\|h\|_\infty}{\omega}\right) \frac{(cR)^2}{\sqrt{(k+1)}} \sqrt{\frac{\frac{1}{R}\|s'\|_\infty}{(cR)^2\omega^2} + c^2\left[1 + \frac{\|s'\|_\infty}{\sqrt{2}}\right]^2} & \text{for } m = 0 \\ \left(1 + \frac{\|h\|_\infty}{\omega}\right) \frac{c^3R^2}{\sqrt{(k+1)|m|}} \left[1 + \frac{\|s'\|_\infty}{\sqrt{|m|(|m|+1)}}\right] & \text{for } |m| \geqslant 1. \end{cases}$$

□

Proof of lemma 3. We have successively

$$\begin{aligned} \langle \mathcal{Y}_{lm}, \Delta^{(m)} \mathcal{Y}_{km} \rangle &= \langle \mathrm{d}\mathcal{Y}_{lm}, \mathrm{d}\mathcal{Y}_{km} \rangle = \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, R^2 \sin\vartheta\, s^2 \\ &\quad \times \left[\frac{1}{R^2} \frac{\partial}{\partial\vartheta} \left(\frac{Y_{lm}^*}{sR}\right) \frac{\partial}{\partial\vartheta} \left(\frac{Y_{km}}{sR}\right) + \frac{m^2 s^{-6}}{R^2 \sin\vartheta^2} \frac{Y_{lm}^*}{R} \frac{Y_{km}}{R} \right] \\ &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \frac{\sin\vartheta}{R^2} \\ &\quad \times \left[\frac{\partial}{\partial\vartheta} Y_{lm}^* \frac{\partial}{\partial\vartheta} Y_{km} - \frac{\partial \ln s}{\partial\vartheta} \frac{\partial}{\partial\vartheta} (Y_{lm}^* Y_{km}) + \left(\frac{\partial \ln s}{\partial\vartheta}\right)^2 Y_{lm}^* Y_{km} \right] \\ &\quad + \frac{1}{R^2} \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \sin(\vartheta) \frac{m^2 s^{-4}}{R^2 \sin^2\vartheta} Y_{lm}^* Y_{km} \\ &= \frac{1}{R^2} \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \sin\vartheta \left[\frac{\partial}{\partial\vartheta} Y_{lm}^* \frac{\partial}{\partial\vartheta} Y_{km} + \frac{m^2 s^{-4}}{R^2 \sin^2\vartheta} Y_{lm}^* Y_{km} \right] \\ &\quad + \frac{1}{R^2} \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\phi\, \sin\vartheta \left[\left(\frac{\partial \ln s}{\partial\vartheta}\right)^2 - \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial \ln s}{\partial\vartheta}\right) \right] \cdot Y_{lm}^* Y_{km} \\ &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \frac{\sin\vartheta}{R^2} \\ &\quad \times \left[m^2 \frac{s^{-4} - 1}{R^2 \sin^2\vartheta} + \left(\frac{\partial \ln s}{\partial\vartheta}\right)^2 - \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial \ln s}{\partial\vartheta}\right) \right] Y_{lm}^* Y_{km}. \end{aligned}$$

It is easy to check that $s(\vartheta)$ is at least of C^2 class, so that the function $h : [0, \pi] \to \mathbb{R}$,

$$h(\vartheta) = m^2 \frac{s^{-4} - 1}{R^2 \sin^2\vartheta} + \left(\frac{\partial \ln s}{\partial\vartheta}\right)^2 - \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial \ln s}{\partial\vartheta}\right)$$

is at least of C^0 class. Finally

$$\begin{aligned} \langle \mathcal{Y}_{lm}, \Delta^{(m)} \mathcal{Y}_{km} \rangle &= \frac{1}{R^2} \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\phi\, \sin\vartheta \cdot h(\vartheta) \cdot Y_{lm}^* Y_{km} \\ &= \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \sin(\vartheta) s^2 \cdot h(\vartheta) \cdot \frac{Y_{lm}^*}{Rs} \frac{Y_{km}}{Rs} = \langle \mathcal{Y}_{lm}, h \cdot \mathcal{Y}_{km} \rangle. \end{aligned}$$

For the second part

$$
\begin{aligned}
|\langle \upsilon | P_k^{(m)} \circ \Delta^{(m)} \circ (I - P_k^{(m)}) | u \rangle| &= |\langle P_k^{(m)} \upsilon | \Delta^{(m)} | (I - P_k^{(m)}) u \rangle| \\
&= |\langle P_k^{(m)} \upsilon, h \cdot (I - P_k^{(m)}) u \rangle| \leqslant \|P_k^{(m)} \upsilon\| \cdot \|h \cdot (I - P_k^{(m)}) u\| \leqslant \|h\|_\infty \|\upsilon\| \cdot \|u\|.
\end{aligned}
$$

□

Proof of lemma 4.

Proposition 5. The application $\tilde{g} : (0, \pi) \times [0, 2\pi] \to M(2 \times 2)$

$$
\tilde{g}(\vartheta, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin(\vartheta)^2 \end{pmatrix}
$$

defines a metric on $\mathbb{M}$. Moreover, $\sqrt{\frac{\det g}{\det \tilde{g}}} = s^2$.

If we consider the spaces of the squared integrable functions with the measures induced by the two metrics, $L_2(\mathbb{M}, \mu_g)$ and $L_2(\mathbb{M}, \mu_{\tilde{g}})$, and the spaces of one-differential forms with the standard scalar products, $A^{(1)}(\mathbb{M}, \mu_g)$ and $A^{(1)}(\mathbb{M}, \mu_{\tilde{g}})$, then we obtain the following proposition.

Proposition 6. The spaces $L_2(\mathbb{M}, \mu_g)$ and $L_2(\mathbb{M}, \mu_{\tilde{g}})$ coincide, as do $A^{(1)}(\mathbb{M}, \mu_g)$ and $A^{(1)}(\mathbb{M}, \mu_{\tilde{g}})$.

Proof. For $f \in L_2(\mathbb{M}, \mu_g)$ we have

$$
\|f\|_{\tilde{g}} = \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \sqrt{\det \tilde{g}(\vartheta)} |f(\vartheta, \varphi)|^2 \leqslant \left\| \frac{\sqrt{\det \tilde{g}(\vartheta)}}{\sqrt{\det g(\vartheta)}} \right\|_\infty \cdot \|f\|_g^2 \leqslant \infty
$$

thus, $f \in L_2(\mathbb{M}, \mu_{\tilde{g}})$. Analogous, for $f \in L_2(\mathbb{M}, \mu_{\tilde{g}})$ results

$$
\|f\|_g = \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \sqrt{\det g(\vartheta)} |f(\vartheta, \varphi)|^2 \leqslant \left\| \frac{\sqrt{\det g(\vartheta)}}{\sqrt{\det \tilde{g}(\vartheta)}} \right\|_\infty \cdot \|f\|_{\tilde{g}}^2 \leqslant \infty
$$

thus $f \in L_2(\mathbb{M}, \mu_g)$.

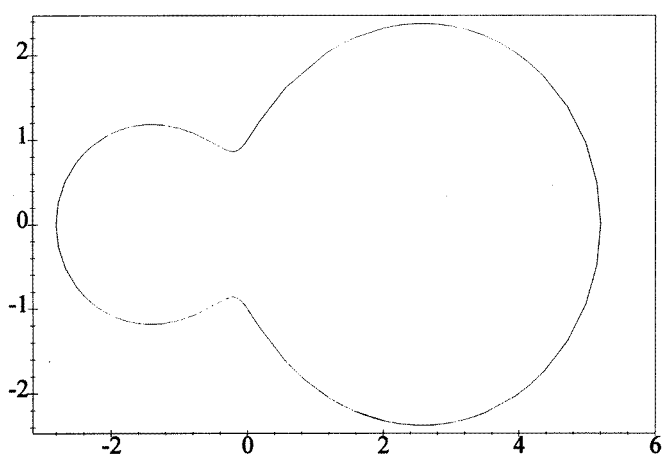

Figure 1. $r(\theta) = 1 + 1.2\cos(\theta) + 3\cos(\theta)^2$.

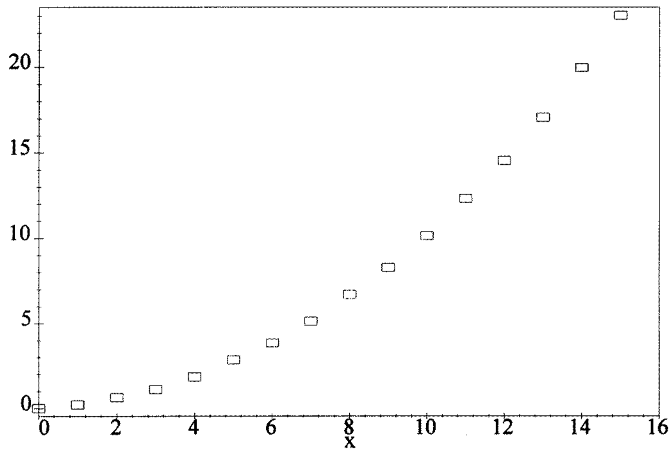

Figure 2. Eigenvalues of the 15×15 truncated matrix.

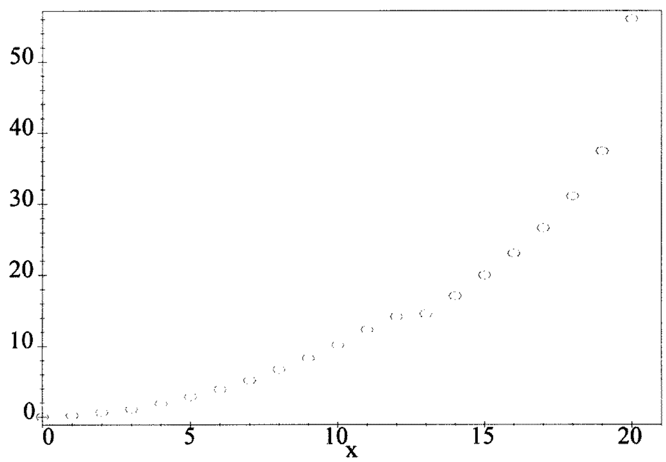

Figure 3. Eigenvalues of the 20×20 truncated matrix.

Let $\omega \in A^{(1)}(\mathbb{M}, \mu_g)$, $\omega = \omega_\vartheta \, \mathrm{d}\vartheta + \omega_\varphi \, \mathrm{d}\varphi$. Following

$$\int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \sqrt{\det \tilde{g}(\vartheta)} g(\bar{\omega}, \omega) = \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \sqrt{\det \tilde{g}(\vartheta)} \left[|\omega_\vartheta|^2 + \frac{|\omega_\varphi|^2}{\det \tilde{g}} \right]$$
$$\leqslant \max \left\{ \sqrt{\frac{\det \tilde{g}}{\det g}}, \sqrt{\frac{\det g}{\det \tilde{g}}} \right\} \cdot \|\omega\|_g^2 \geqslant \infty$$

thus $\omega \in A^{(1)}(\mathbb{M}, \mu_{\tilde{g}})$. The same steps can be followed to show that $\omega \in A^{(1)}(\mathbb{M}, \mu_{\tilde{g}}) \Rightarrow \omega \in A^{(1)}(\mathbb{M}, \mu_g)$. Denoting

$$c = \sqrt{\max\{\|s^2\|_\infty, \|s^{-2}\|_\infty\}}$$

we have on $L_2(\mathbb{M}, \mu_g) \equiv L_2(\mathbb{M}, \mu_{\tilde{g}})$:

$$\frac{1}{c} \| \ \|_{\tilde{g}} \leqslant \| \ \|_g \leqslant c \| \ \|_{\tilde{g}}$$

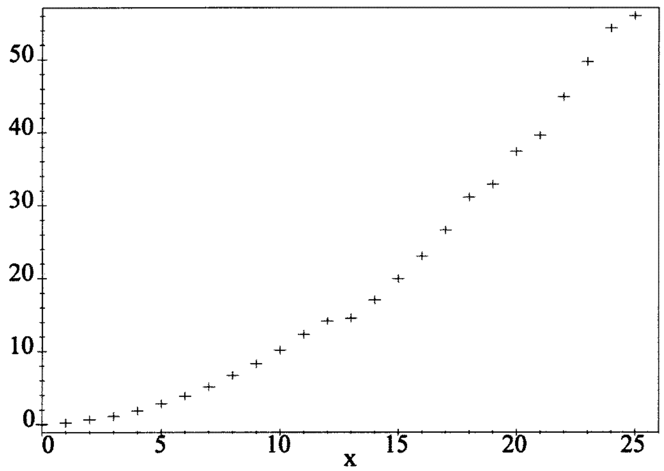

Figure 4. Eigenvalues of the 25×25 truncated matrix.

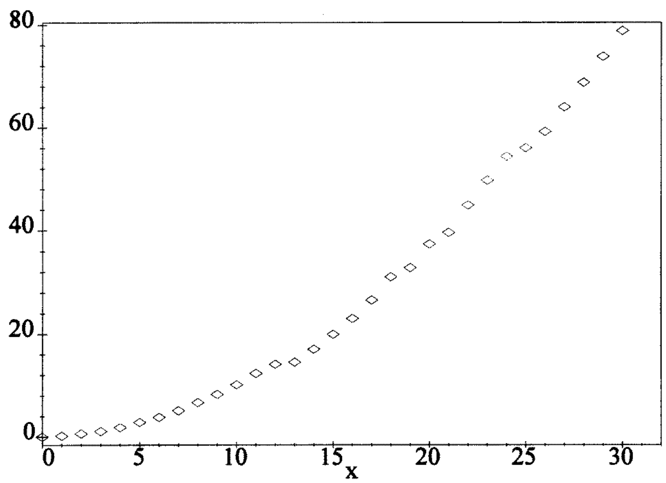

Figure 5. Eigenvalues of the 30×30 truncated matrix.

and, on $A^{(1)}(\mathbb{M}, \mu_{\tilde{g}}) = A^{(1)}(\mathbb{M}, \mu_g)$:

$$\frac{1}{c}\| \ \|_{\tilde{g}} \leqslant \| \ \|_g \leqslant c\| \ \|_{\tilde{g}}.$$

$\square$

Now, we have successively

$$|\langle \mathcal{Y}_{lm} | \upsilon_n^{(m)} \rangle_g| = \left| \left\langle \frac{Y_{lm}}{sR} | \upsilon_n^{(m)} \right\rangle_g \right| = \left| \left\langle \frac{Y_{lm}}{R} | s^{-1} \cdot \upsilon_n^{(m)} \right\rangle_g \right|$$
$$= \frac{\left| \left\langle \tilde{\Delta} \frac{Y_{lm}}{R} | s^{-1} \cdot \upsilon_n^{(m)} \right\rangle_g \right|}{\frac{l(l+1)}{R^2}} = \frac{\left| \left\langle \tilde{\Delta} \frac{Y_{lm}}{R} | s \cdot \upsilon_n^{(m)} \right\rangle_{\tilde{g}} \right|}{\frac{l(l+1)}{R^2}} \leqslant \frac{\left\| \mathrm{d} \frac{Y_{lm}}{R} \right\|_{\tilde{g}} \cdot \| \mathrm{d}(s \cdot \upsilon_n^{(m)}) \|_{\tilde{g}}}{\frac{l(l+1)}{R^2}}$$

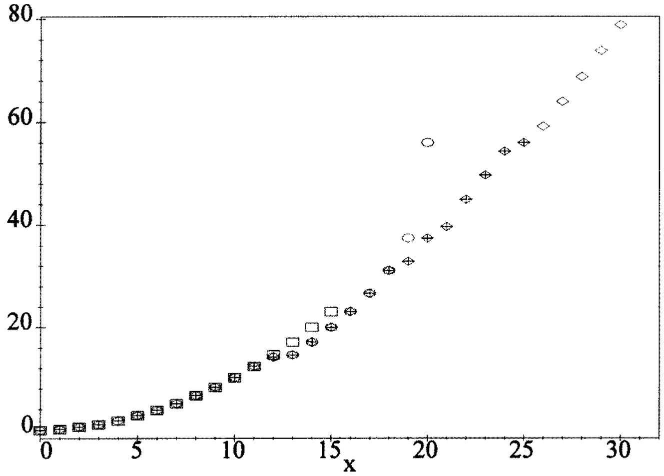

Figure 6. The superposition.

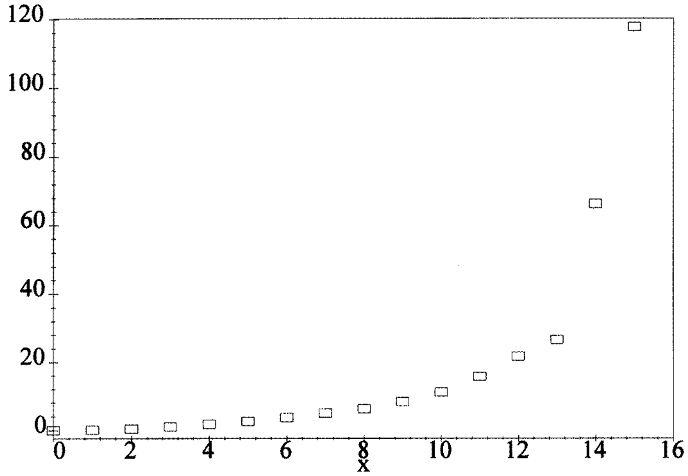

Figure 7. Eigenvalues of the 15×15 truncated matrix.

$$\leqslant \frac{R}{\sqrt{l(l+1)}}[\|\mathrm{d}s\|_\infty \|\upsilon_n^{(m)}\|_{\tilde{g}} + \|s\|_\infty \|\mathrm{d}\upsilon_n^{(m)}\|_{\tilde{g}}] \leqslant \frac{cR\left[\|\mathrm{d}s\|_\infty + c\sqrt{\lambda_n^{(m)}}\right]}{\sqrt{l(l+1)}}.$$

For the second set of inequalities:

$$|\langle \mathcal{Y}_{lm}|\upsilon_n^{(m)}\rangle| = \frac{1}{\lambda_n^{(m)}}|\langle \mathcal{Y}_{lm}|\Delta\upsilon_n^{(m)}\rangle| \leqslant \frac{1}{\lambda_n^{(m)}}\|\mathrm{d}\mathcal{Y}_{lm}\|_g \|\mathrm{d}\upsilon_n^{(m)}\|_g \leqslant \frac{c\|\mathrm{d}\mathcal{Y}_{lm}\|_{\tilde{g}}}{\sqrt{\lambda_n^{(m)}}}$$

$$\leqslant \frac{c\left[\|\mathrm{d}s^{-1}\|_\infty + \frac{c}{R}\sqrt{l(l+1)}\right]}{\sqrt{\lambda_n^{(m)}}}.$$

For the last set of inequalities of lemma 4, once we have the results of the last proposition we can follow the method of [2], or that presented in [3]. □

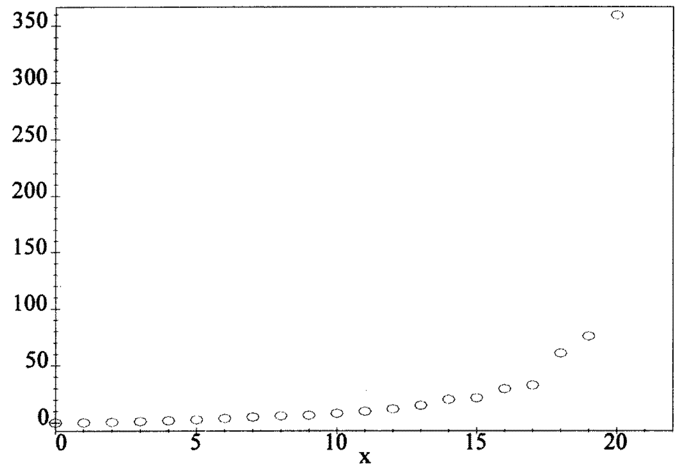

Figure 8. Eigenvalues of the 20×20 truncated matrix.

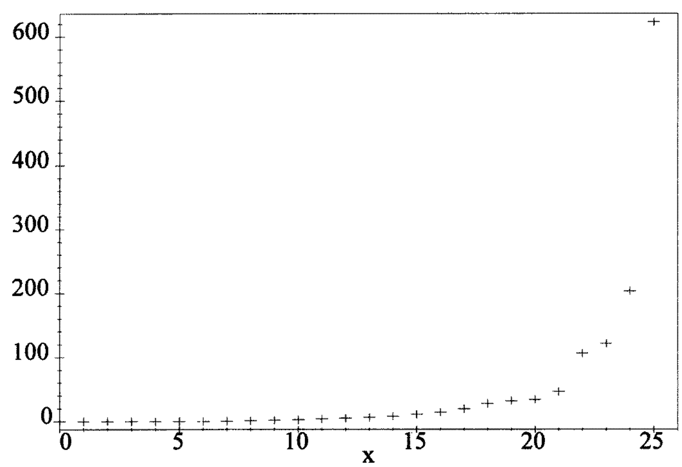

Figure 9. Eigenvalues of the 25×25 truncated matrix.

3. Numerical application

Generally, the spectrum of the truncated matrices does not converge at the exact spectrum. Without additional results, one knows that only the lowest eigenvalue of the truncated matrices converges at the exact value [6]. The results of section 2 have another important consequence: in the proposed basis, the spectrum of the truncated matrices converges at the exact spectrum. Moreover, because the matrix of the Laplace–Beltrami operator in the $\mathcal{Y}$ basis is 'quasidiagonal' in the sense that all nondiagonal elements are bounded by $\|h\|_\infty$ and the diagonal elements increase approximately as $l(l+1)/R^2$, it is to be expected that the spectrum of these truncated matrices is very stable. That means, that even for low dimensions these matrices give us a good approximation of the exact spectrum. Let us choose the particular surfaces of figure 1 for our numerical application.

The eigenvalue for different truncated matrices and $m = 0$ are presented in figures 2–6.

Now, let us choose an orthonormal basis for which the affirmation of lemma 4 is not

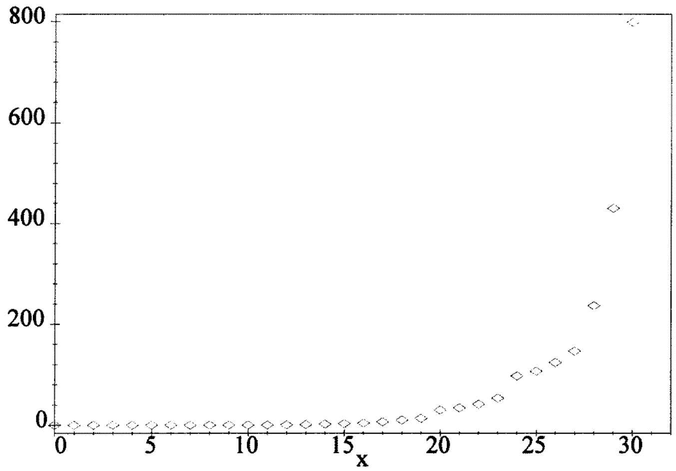

Figure 10. Eigenvalues of the 30×30 truncated matrix.

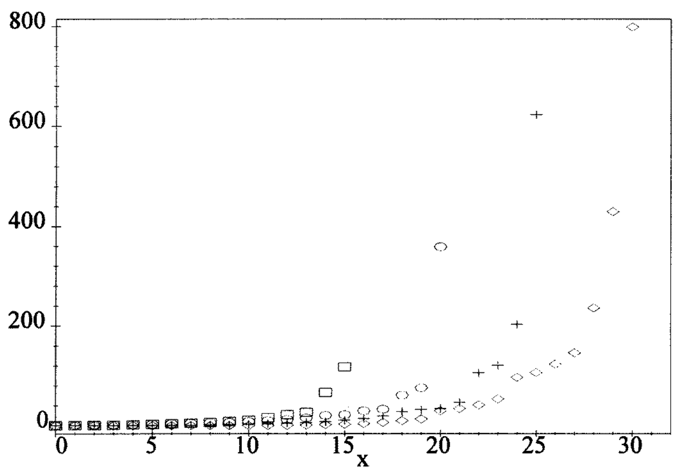

Figure 11. The superposition.

true. If $d\mu_g(\theta, \phi) = \sigma(\theta, \phi) \sin\theta \cdot d\theta\, d\phi$ is the measure induced by the metric g in the coordinates $\{\theta, \phi\}$, then:

Proposition 7. The set $\tilde{\mathcal{Y}}$ of functions:

$$\tilde{\mathcal{Y}}_{lm}(\theta, \phi) = \frac{Y_{lm}(\theta, \phi)}{\sqrt{\sigma(\theta, \phi)}} \qquad m = 0, 1, \ldots, \; l = |m|, |m| + 1, \ldots$$

is an orthonormal basis in $L_2(\mathbb{M}, \mu_g)$.

The proof of this proposition is analogous to that of proposition 1. The eigenvalues of different truncated matrices, calculated in this basis and for the case $m = 0$, are presented in figures 7–11.

The numerical application shows that in this case the spectrum of the truncated matrices is very unstable. This instability can be considered as an indicator of the fact that for the $\tilde{\mathcal{Y}}$ basis the affirmation of our theorem is not true.

4. Conclusion

This paper has shown how to construct an orthonormal basis in the space of square integrable functions defined on a C^∞ surface with axial symmetry, a basis which is appropriate for the problems which involve the Laplace–Beltrami operator. The procedure is standard, in the sense that it can be applied following the same steps for any C^∞ surface with axial symmetry. The stability of the truncated matrices spectrum was theoretically anticipated and numerically verified. By a practical point of view, this allows us to use truncated matrices with small number of rows and columns.

References

[1] Prodan E, Gheorghiu E and Vranceanu D 1996 The dielectric behavior of suspensions of arbitrarily shaped, shelled, charged particles *Proc. New Trends in Biotechnologies (Bucharest)*
[2] Bando S and Urakawa H 1983 Generic properties of the eigenvalues of the Laplacian for compact Riemannian manifolds *Tohoku Math. J.* **35** 155–72
[3] Craioveanu M and Puta M 1988 *Introduction in Spectral Geometry* (Bucharest: Romanian Academy)
[4] Narasimhan R 1968 *Analysis on Real and Complex Manifolds* (Amsterdam: North-Holland)
[5] Reed M and Simon B 1980 *Functional Analysis* (New York: Academic) p 286
[6] Reed S and Simon B 1978 *Analysis of Operators* (New York: Academic) p 79